Indian Statistical Institute, Bangalore Centre
B.Math.(Hons.) I year, First Semester

Semesteral Examination (Back Paper)
Analysis I
Instructor: Pl Muthuramalingam

Date: January 13, 2010

Maxmimum marks you can get is 40.

1. Give an example of a bounded sequence which is not a cauchy sequence and prove your claim. [1]

2. Let $g : (0,1] \longrightarrow R$ be the continuous function given by $g(x) = \sin\left(\frac{1}{x}\right)$. Let $f : [0,1] \longrightarrow R$ be any function such that $f(x) = g(x)$ for all x in $(0,1]$. Show that f can not be continuous. [2]

3. Let $f : [0,1] \longrightarrow R$ be given by $f(x) = x$ if x is rational and $1 - x$ if x is irrational. Show that f is continuous at $\frac{1}{2}$. [2]

4. Give an example of a continuous function $f : (0,1] \longrightarrow R$ which is not bounded above. [1]

5. Let $I_1 \supset I_2 \supset \cdots$ be a sequence of closed intervals with length $I_r \longrightarrow 0$ as $r \longrightarrow \infty$. If $x_j, \epsilon\ I_j$ then show that the sequence $\{x_1,\ x_2,\ x_3, \cdots\}$ is a cauchy sequence. [2]

6. (a) Let $a_n > 0, \sum a_n^2 \ < \infty,\ \partial > \frac{1}{2}$. Then shwo that $\sum\limits_1^\infty \frac{a_n}{n^\partial}$ exists. [2]

   (b) Let $\mathbb{B} > \frac{1}{2}$. Show that $\sum\ b_n < \infty$ where $b_n = \frac{\sqrt{n+1}-\sqrt{n}}{n^{\mathbb{B}}}$. [2]

7. (a) Show that $f : R \longrightarrow R$ given by $f(x) = x^2$ is not uniformly continuous. [2]

   (b) Give an example of uniformly continuous functions g_1, g_2 such that the product $g_1\ g_2$ is not uniformly continuous and prove your claim.[1]

   (c) Let $g : J \longrightarrow R$ be uniformly continuous.Show that if $x_1, x_2, \cdots$ is a cauchy sequence in J, then $g(x_1), \cdots$ is a cauchy sequence. [2]

   (d) Let $k : J \longrightarrow R$ be a continuous, differentiable function and the derivative be bounded and continuous. Show that k is uniformly continuous. Here J is a bounded or unbounded interval. [2]

8. If $a_1, a_2, \cdots$ is a sequence of reals with $\sum |a_n| < \infty$, then $\sum a_n$ exists.

9. Let $a_1, a_2, a_3, \cdots$ be a sequence of reals. $s_n = a_1 + a_2 + a_3 + ...a_n$. Assume that the sequence s_{3n} is convergent. Then $\sum a_n$ exists $\Leftrightarrow$ $a_r \longrightarrow 0$ as $r \longrightarrow \infty$. [2]

10. Let $f : [0, 1] \longrightarrow [0, \infty)$ be any function assume that there exists $M \geq 0$ such that for all subsets $\{x_1, x_2, \cdots, x_k\}$ of $[0, 1]$, one has

    $f(x_1) + f(x_2) + \cdots + f(x_k) \leq M$. Show that

    $G = \{x : f(x) \neq 0\}$ is a countable set. [2]

11. Let $g : [0, 1] \longrightarrow R$ be any continous function with $g(0) < 0 < g(1)$. Show that g assumes the value 0. [3]

12. Let $f : [a, b] \longrightarrow R$ be continuous, differentiable and f' be continuous. Show that $\lim\limits_{\delta \longrightarrow 0} \sup\limits_{a \leq x \leq b} \sup\limits_{0 < |t-x| \leq \delta} \left| \frac{f(t)-f(x)}{t-x} - f'(x) \right| = 0$ [2]

13. Let a and b be real numbers. If the series $(a+b)+(a+2b)+(a+3b)+\cdots$ is convergent, then show that $b = 0$ and $a = 0$. [2]

14. Let A, B be bounded subsets of $[0, \infty)$. Let $C = \{ab : a \in A, b \in B\}$. Let $x = \sup A, y = \sup B, z = \sup C$. Note that x need not be in A and y need not be in B. Show that $z = xy$. [2]

15. (a) Let $a_1, a_2, \cdots$ be a sequence with $a_j \geq 0$. Let $\sum\limits_{1}^{\infty} a_j$ be convergent let $n_1 < n_2 < n_3 < \cdots$ be increasing sequence of natural numbers let $b_j = a_{n_j} \cdots$. Show that $\sum b_j$ is convergent. [2]

    (b) Give an example of a real sequence $x_1, x_2, \cdots$ and a subsequence $x_{n_2}, x_{n_2} \cdots$ such that $\sum x_j$ is convergent and $\sum x_{n_j}$ is not convergent prove your claim.

16. Define $f : [0, 1] \longrightarrow R$ by $f(x) = x^a \sin(\frac{1}{x^c})$, where $c > o$ and $a \geq 0$ for $x > 0$, $f(0) = 0$.

    (i) If f is continuous at 0, show that $a > 0$. [2]

17. Let $a_1, a_2, \cdots$ be a sequence of reals with $\sum a_j$ convergent. Let $n_1 < n_2 < n_3 < \cdots$.

    Put $b_1 = a_1 + a_2 + \cdots + a_{n_1}$,

    $b_2 = a_{n_1+1} + a_{n_1+2} + \cdots + a_{n_2}$.

    $b_3 = a_{n_2+1} + a_{n_2+2} + \cdots + a_{n_3}, etc.$

    Show that the series $b_1 + b_2 + \cdots$ is convergent and converges to $\sum a_r$.

    [2]

18. Show that any disjoint collection of bounded intervals each of positive length is finite or countable.